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Regularized tripartite continuous variable EPR-type states with Wigner functions and CHSH violations

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Abstract
We consider tripartite entangled states for continuous variable systems of EPR type, which generalize the famous bipartite CV EPR states (eigenvectors of conjugate choices $X_1 - X_2$, $P_1 + P_2$, of the systems' relative position and total momentum variables). We give the regularized forms of such tripartite EPR states in second-quantized formulation, and derive their Wigner functions. This is directly compared with the established NOPA-like states from quantum optics. Whereas the multipartite entangled states of NOPA type have singular Wigner functions in the limit of large squeezing, $r \to \infty$, or $\tanh r \to 1^-$ (approaching the EPR states in the bipartite case), our regularized tripartite EPR states show singular behaviour not only in the approach to the EPR-type region ($s \to 1$ in our notation), but also for an additional, auxiliary regime of the regulator ($s \to \sqrt{2}$). While the $s \to 1$ limit pertains to tripartite CV states with singular eigenstates of the relative coordinates and remaining squeezed in the total momentum, the $s \to \sqrt{2}$ limit yields singular eigenstates of the total momentum, but squeezed in the relative coordinates. Regarded as expectation values of displaced parity measurements, the tripartite Wigner functions provide the ingredients for generalized CHSH inequalities. Violations of the tripartite CHSH bound ($B_3 \leqslant 2$) are established, with $B_3 \cong 2.09$ in the canonical regime ($s \to 1^+$), as well as $B_3 \cong 2.32$ in the auxiliary regime ($s \to \sqrt{2^+}$).

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1. Introduction

The nature of quantum entanglement has been pursued almost since the inception of quantum mechanics itself. Whereas the early insights of Schrödinger, as well as of Einstein, Podolsky and Rosen (EPR) [1] were framed in the *Gedankenexperiment* mode of discussion with continuous degrees of freedom (particle position and momentum eigenstates), the issues were taken up quantitatively in Bell's theorem [2] for the case of spin degrees of freedom, via the transcription to this context given by Bohm [3]. However, continuous variable (CV) systems are the natural framework for most quantum optics and quantum communication work [4, 5], and Bell, and the more general Clauser, Horne, Shimony and Holt (CHSH) inequalities [6], are important measures of entanglement. A technical difficulty in working with continuous variables is that the theoretical ideal EPR type states are singular, whereas experimental investigations require regularized states. In the bipartite case these may be provided by so-called NOPA squeezed states [7–11] in the large squeezing limit. In an alternative approach, Fan and Klauder [12] constructed somewhat more general classes of EPR states, but without providing a regularization.

In the extension to multipartite cases, a natural question is the choice of relative variables, chosen from amongst the positions and momenta of the constituent particles of the system, which will provide an appropriate generalization of the bipartite EPR states (with or without regularization)[3]. One possibility is provided by the multipartite Greenberger–Horne–Zeilinger NOPA-like squeezed states [13, 14]. These have the virtue of experimental accessibility [8, 9], and do show singular behaviour in the large squeezing limit which moreover leads to violations of the multipartite CHSH inequalities [15]. For a derivation of the CHSH inequalities for N-particle systems see for example [16]. However, many other choices of relative variables exist—see for example [17] and references therein.

In this paper we take up a logical generalization of the original EPR suggestion, in selecting simultaneously diagonalizable joint degrees of freedom from amongst the canonical Jacobi relative coordinates of the particles. Section 2 is divided into two subsections. In the first we introduce the bipartite Fan and Klauder EPR-like state and propose a possible regularization. This is shown to be identical to the bipartite NOPA state (which approximates the ideal EPR limit) with squeezing parameter $r \to \infty$. In fact the correspondence is that our regularization $s \to 1^+$ coincides with $\tanh r \to 1^-$ as $r \to \infty$, with $\tanh r = 1/s^2$. In the bipartite case the Fan and Klauder states are more general than the original EPR states in that they realize explicit nonzero eigenvalues of total or relative positions and momenta; but a study of the Wigner functions reveals that such nonzero eigenvalues can be absorbed into shifts of the complex displacement parameters, and so in the bipartite case the NOPA states do not lose any generality in not allowing for such nonzero eigenvectors. In the next subsection we follow the Fan–Klauder approach, in second quantized formalism, to derive the explicit theoretical tripartite CV states of EPR type conforming to this structure, and we develop the methods to provide a plausible regularization.

In section 3, we derive explicit Wigner functions for our regularized tripartite states of EPR type by interpreting the Wigner functions themselves as expectation values of displaced parity measurement operators. As could be expected from their different second-quantized forms, the tripartite EPR and NOPA Wigner functions differ significantly (the appendix provides a comparison of the second-quantized forms and their respective Wigner functions, including an explicit evaluation of the former in the NOPA-like case, and a detailed derivation of the latter

[3] We refer conventionally to the subsystems as 'particles', but it should be borne in mind that the CV systems could equally be independent photon polarization modes, photon modes or even joint photon and phonon degrees of freedom.

for our EPR-type states). Specifically, whereas the multipartite NOPA Wigner functions are singular in the large squeezing limit, our regularized tripartite states of EPR type admit two different singular regimes: not only in the EPR-type regime ($s \to 1$ in our notation), where of course the Wigner function still differs from that of NOPA, but also, rather unexpectedly, for an additional, auxiliary regime of the regulator ($s \to \sqrt{2}$). In section 4 we exploit the fact that Wigner functions are immediately applicable as summands in the appropriate tripartite CHSH inequalities. We explore the two singular regimes and their Bell operator expectation values which control the classical–quantum boundary via the CHSH bound ($B_3 \leqslant 2$) and identify some instances of violations for each of the cases $s \to 1^+$ and $s \to \sqrt{2^+}$. Section 5 includes further discussion of our findings, as well as comparison with the recent work [18, 19], which provides a general construction of ideal EPR states, and some concluding remarks.

2. Regularized CV EPR states

2.1. Bipartite states

The case considered by EPR in [1] discusses the simultaneous diagonalization of the two commuting variables of difference in position $(X_1 - X_2)$ and total momentum $(P_1 + P_2)$, where $X_j, P_j, j = 1, 2$ are a standard pair of canonically conjugate variables with $[X_j, P_k] = \mathrm{i}\delta_{jk}$. Fan and Klauder [12] give an explicit form for the common eigenvectors of the relative position and total momentum for two EPR particles in terms of creation operators as follows:

$$|\eta\rangle = \mathrm{e}^{-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger}|00\rangle, \tag{1}$$

where $\eta = \eta_1 + \mathrm{i}\eta_2$ is an arbitrary complex number, $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$ and $|00\rangle \equiv |0, 0\rangle$, the two-mode vacuum state. Thus

$$(X_1 - X_2)\,|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \qquad (P_1 + P_2)\,|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle, \tag{2}$$

with the coordinate and momentum operators definable as

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{2}}(a + a^\dagger), \qquad & X_2 &= \frac{1}{\sqrt{2}}(b + b^\dagger), \\ P_1 &= \frac{1}{\mathrm{i}\sqrt{2}}(a - a^\dagger), \qquad & P_2 &= \frac{1}{\mathrm{i}\sqrt{2}}(b - b^\dagger). \end{aligned} \tag{3}$$

As a genuine representation of ideal generalized EPR states, with appropriate orthonormality and completeness, $|\eta\rangle$ is singular. In this paper we consider the following regularized version:

$$|\eta\rangle_s := N_2\, \mathrm{e}^{-\frac{1}{2s^2}|\eta|^2 + \frac{1}{s}\eta a^\dagger - \frac{1}{s}\eta^* b^\dagger + \frac{1}{s^2}a^\dagger b^\dagger}|00\rangle, \tag{4}$$

with normalization $|N_2|^2 = |(s^4 - 1)|^{1/2}/s^2$.

The bipartite CV state (4) is to be compared with a regularized EPR-like state which has already appeared in the literature—the so-called NOPA state from quantum optics [7–11]:

$$|\mathrm{NOPA}\rangle = \mathrm{e}^{r(a^\dagger b^\dagger - ab)}|00\rangle. \tag{5}$$

NOPA states are produced by Nondegenerate Optical Parametric Amplification, and are the optical analogue to the EPR state in the limit of strong squeezing. The NOPA state has already been shown to be a genuinely entangled state that produces violations of the CHSH inequality [9, 10, 15].

Following [20] on reordering SU(1, 1) operators, we can reorder the expression for NOPA (5) into the following form:

$$\begin{aligned}|\mathrm{NOPA}\rangle &= \mathrm{e}^{r(a^\dagger b^\dagger - ab)}|00\rangle \\ &= \mathrm{e}^{ra^\dagger b^\dagger}\,\mathrm{e}^{-2\ln\cosh(r)\frac{1}{2}(a^\dagger a + b^\dagger b + 1)}\,\mathrm{e}^{-rab}|00\rangle \\ &= \sqrt{1-\tanh^2 r}\,\mathrm{e}^{\tanh r a^\dagger b^\dagger}|00\rangle.\end{aligned} \tag{6}$$

Note from (6) that in the number basis, as $\tanh r \to 1$ the $|\mathrm{NOPA}\rangle$ state approximates the ideal EPR limit:

$$\lim_{r\to\infty}|\mathrm{NOPA}\rangle \approx |\mathrm{EPR}\rangle \approx |0,0\rangle + |1,1\rangle + |2,2\rangle + \cdots. \tag{7}$$

In appendix A.2 it is argued that taking $\eta = 0$ corresponds to a shift in the parameters of the displacement operators (with some constraints on the choice of new parameters), such that we may rearrange the $|\eta\rangle_s$ regularization to show that it approaches the NOPA regularization, with $\tanh r = 1/s^2$. For $\eta = 0$ we therefore have

$$|\eta = 0\rangle_s = N_2\,\mathrm{e}^{\frac{1}{s^2}a^\dagger b^\dagger}|00\rangle. \tag{8}$$

2.2. *Tripartite states*

A suitable analogue of equations (4) or (6) which has the features required of an entangled state, which we analyse in detail below, is defined by

$$|\eta,\eta',\eta''\rangle_s = N_3\,\mathrm{e}^{-\frac{1}{4s^2}|\eta|^2 - \frac{1}{4s^2}|\eta'|^2 - \frac{1}{4s^2}|\eta''|^2 + \frac{1}{s}(\eta a^\dagger + \eta' b^\dagger + \eta'' c^\dagger) + \frac{1}{s^2}(a^\dagger b^\dagger + a^\dagger c^\dagger + b^\dagger c^\dagger)}|000\rangle, \tag{9}$$

with normalization $|N_3|^2 = |(s^4-1)^2(s^4-4)|^{1/2}/s^6$. For the case $\eta = \eta' = \eta'' = 0$, the tripartite EPR-like state becomes

$$|\eta = \eta' = \eta'' = 0\rangle_s = N_3\,\mathrm{e}^{\frac{1}{s^2}(a^\dagger b^\dagger + a^\dagger c^\dagger + b^\dagger c^\dagger)}|000\rangle. \tag{10}$$

Note here that, while the set of states (10) belong to the well-known pure, fully symmetric three-mode Gaussian states, the more general case of (9) where the parameters η, η' and η'' are retained is not symmetric, since the parameters can all differ. For discussion of Gaussian states in relation to entanglement in CV systems, see [21] and references therein.

Whereas the bipartite state $|\eta\rangle_s$ was a simultaneous eigenstate of $(X_1 - X_2)$ and $(P_1 + P_2)$, in the tripartite case the choice of relative variables is no longer immediately apparent. In a similar manner to the derivation of (4), it is readily established using manipulations of the type

$$a\,\mathrm{e}^A = \mathrm{e}^A\left\{a - [A,a] + \tfrac{1}{2}[A,[A,a]] + \cdots\right\}, \tag{11}$$

that generically $|\eta,\eta',\eta''\rangle_s$ is an eigenstate of the following combinations:

$$\begin{aligned}\left(a - \frac{1}{s^2}(b^\dagger + c^\dagger)\right)|\eta,\eta',\eta''\rangle_s &= \frac{1}{s}\eta|\eta,\eta',\eta''\rangle_s,\\ \left(b - \frac{1}{s^2}(c^\dagger + a^\dagger)\right)|\eta,\eta',\eta''\rangle_s &= \frac{1}{s}\eta'|\eta,\eta',\eta''\rangle_s,\\ \left(c - \frac{1}{s^2}(a^\dagger + b^\dagger)\right)|\eta,\eta',\eta''\rangle_s &= \frac{1}{s}\eta''|\eta,\eta',\eta''\rangle_s.\end{aligned} \tag{12}$$

From this it is clear that different values of s will dictate limiting cases wherein $|\eta,\eta',\eta''\rangle_s$ becomes a singular eigenvalue of various choices of relative variables. (Note that in the bipartite case we could have introduced $|\eta\rangle_s$ as $|\eta, \eta'\rangle_s$ analogously, recovering (4) in the case $\eta' = -\eta^*$.) Keeping s general, the eigenvalue equations become

$$\frac{1}{\sqrt{2}}\left(s+\frac{1}{s}\right)(X_1 - X_2) + \frac{\mathrm{i}}{\sqrt{2}}\left(s-\frac{1}{s}\right)(P_1 - P_2)|\eta, \eta', \eta''\rangle_s = (\eta - \eta')|\eta, \eta', \eta''\rangle_s,$$
$$\frac{1}{\sqrt{2}}\left(s+\frac{1}{s}\right)(X_2 - X_3) + \frac{\mathrm{i}}{\sqrt{2}}\left(s-\frac{1}{s}\right)(P_2 - P_3)|\eta, \eta', \eta''\rangle_s = (\eta' - \eta'')|\eta, \eta', \eta''\rangle_s,$$
$$\frac{1}{\sqrt{2}}\left(s-\frac{2}{s}\right)(X_1 + X_2 + X_3) + \frac{\mathrm{i}}{\sqrt{2}}\left(s+\frac{2}{s}\right)(P_1 + P_2 + P_3)|\eta, \eta', \eta''\rangle_s$$
$$= (\eta + \eta' + \eta'')|\eta, \eta', \eta''\rangle_s. \qquad (13)$$

From (13) it is clear that the singular cases will occur for $s = 1$ and $s = \sqrt{2}$. For the case $s = 1$ we evidently have a singular eigenstate of the relative coordinates, while remaining a *squeezed* state [5] of the total momentum. Conversely, for $s = \sqrt{2}$ we have a singular eigenstate of the total momentum, but a squeezed state of the relative coordinates.

If we construct mode operators corresponding to the Jacobi relative variables and the canonical centre-of-mass variables, say

$$\begin{aligned}
\mathfrak{a}_{\mathrm{rel}} &= \frac{1}{2}(X_1 - X_3) + \frac{\mathrm{i}}{2}(P_1 - P_3),\\
\mathfrak{b}_{\mathrm{rel}} &= \frac{1}{2\sqrt{3}}(X_1 + X_3 - 2X_2) + \frac{\mathrm{i}}{2\sqrt{3}}(P_1 + P_3 - 2P_2),\\
\mathfrak{a}_{\mathrm{cm}} &= \frac{1}{\sqrt{6}}(X_1 + X_2 + X_3) + \frac{\mathrm{i}}{\sqrt{6}}(P_1 + P_2 + P_3),
\end{aligned} \qquad (14)$$

then we find, from (13) for general s:

$$\begin{aligned}
\left(s\mathfrak{a}_{\mathrm{rel}} + \frac{1}{s}\mathfrak{a}_{\mathrm{rel}}^\dagger\right)|\eta, \eta', \eta''\rangle_s &= \frac{1}{\sqrt{2}}(\eta - \eta'')|\eta, \eta', \eta''\rangle_s,\\
\left(s\mathfrak{b}_{\mathrm{rel}} + \frac{1}{s}\mathfrak{b}_{\mathrm{rel}}^\dagger\right)|\eta, \eta', \eta''\rangle_s &= \frac{1}{\sqrt{6}}(\eta - 2\eta' + \eta'')|\eta, \eta', \eta''\rangle_s,\\
\left(s\mathfrak{a}_{\mathrm{cm}} - \frac{2}{s}\mathfrak{a}_{\mathrm{cm}}^\dagger\right)|\eta, \eta', \eta''\rangle_s &= \frac{1}{\sqrt{3}}(\eta + \eta' + \eta'')|\eta, \eta', \eta''\rangle_s,
\end{aligned} \qquad (15)$$

from which it is again obvious that for $s = 1$ or $s = \sqrt{2}$, canonical combinations arise in the first two, and last cases respectively. On the other hand, the non-canonical combinations appearing for $s = 1$ in the third, and $s = \sqrt{2}$ in the first two cases, indicate that the squeezing parameters have the values $\frac{1}{2}\ln 3$ in each instance.

Having established the structure of the tripartite EPR-like states, we can examine their behaviour when applied to Wigner functions, and the consequences of using these states in CHSH inequalities.

3. Tripartite Wigner function

The Wigner function [22, 23], was an attempt to provide the Schrödinger wavefunction with a probability in phase space. The time-independent function for one pair of x and p variables is

$$W(x, p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\, \psi^*(x+y)\psi(x-y)\,\mathrm{e}^{2\mathrm{i}py/\hbar}. \qquad (16)$$

Alternatively, it has been shown [24, 25] that a useful expression of the Wigner function is in the form of quantum expectation values. For N modes, the Wigner function for a state $|\psi\rangle$

may be expressed as the expectation value of the displaced parity operator, where the parity operator itself performs reflections about phase-space points (α_j), where $\alpha_j = \frac{1}{\sqrt{2}}(x_j + \mathrm{i}p_j)$, with $j = 1, 2, \ldots, N$ denoting the mode, and

$$W(\alpha_1, \alpha_2, \ldots, \alpha_N) = \left(\frac{2}{\pi}\right)^N \langle \Pi(\alpha_1, \alpha_2, \ldots, \alpha_N)\rangle. \tag{17}$$

The displaced parity operator is

$$\Pi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \otimes_{\boldsymbol{j=1}}^{\boldsymbol{N}} D_j(\alpha_j)(-1)^{n_j} D_j^\dagger(\alpha_j), \tag{18}$$

where n_j are the number operators, and for each mode the Glauber displacement operators are of the form

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}. \tag{19}$$

When we express the Wigner function in the form of (17), we can derive a set of these functions to construct the inequalities that will be discussed in section 4. The tripartite Wigner function for $|\eta, \eta', \eta''\rangle_s$ becomes

$$\begin{aligned} W(\alpha, \beta, \gamma) = {} & \left(\frac{2}{\pi}\right)^3 N_3^2\, \mathrm{e}^{-\frac{1}{2s^2}|\eta|^2 - \frac{1}{2s^2}|\eta'|^2 - \frac{1}{2s^2}|\eta''|^2} \\ & \times \langle 000|\exp\left(\frac{1}{s}(\eta^* a + \eta'^* b + \eta''^* c) + \frac{1}{s^2}(ab + ac + bc)\right) \\ & \times \mathrm{e}^{\alpha a^\dagger - \alpha^* a}\, \mathrm{e}^{\beta b^\dagger - \beta^* b}\, \mathrm{e}^{\gamma c^\dagger - \gamma^* c} (-1)^{n_a + n_b + n_c}\, \mathrm{e}^{\alpha^* a - \alpha a^\dagger}\, \mathrm{e}^{\beta^* b - \beta b^\dagger}\, \mathrm{e}^{\gamma^* c - \gamma c^\dagger} \\ & \times \exp\left(\frac{1}{s}(\eta a^\dagger + \eta' b^\dagger + \eta'' c^\dagger) + \frac{1}{s^2}(a^\dagger b^\dagger + a^\dagger c^\dagger + b^\dagger c^\dagger)\right)|000\rangle. \end{aligned} \tag{20}$$

We evaluate such matrix elements by commuting mode operators with the parity operator and rearranging using BCH identities before casting the operators into anti-normal ordered form. Then a complete set of coherent states is inserted and integrated over. As indicated in section 2.1 and shown in appendix A.2, we can absorb the η, η', η'' parameters by shifting the displacement parameters up to a factor: $W_{\eta,\eta',\eta''}(\alpha', \beta', \gamma') = E(\alpha', \beta', \gamma', \eta, \eta', \eta'')W_{0,0,0}(\alpha, \beta, \gamma)$. We are free to choose instances where $E(\alpha', \beta', \gamma', \eta, \eta', \eta'') = 1$, and henceforth we assume $\eta = \eta' = \eta'' = 0$ unless otherwise stated, and write simply $W(\alpha, \beta, \gamma)$. With this shift in displacements, the Wigner function for our tripartite state becomes

$$\begin{aligned} W(\alpha, \beta, \gamma) = {} & \frac{8}{\pi^3}\exp\left(\frac{1}{(s^4-4)(s^4-1)}[C_1(|\alpha|^2 + |\beta|^2 + |\gamma|^2) \right.\\ & + C_2(\alpha\beta + \alpha\gamma + \beta\gamma + \alpha^*\beta^* + \alpha^*\gamma^* + \beta^*\gamma^*) \\ & + C_3(\alpha\beta^* + \alpha\gamma^* + \beta\alpha^* + \beta\gamma^* + \gamma\alpha^* + \gamma\beta^*) \\ & \left. + C_4(\alpha^2 + \beta^2 + \gamma^2 + \alpha^{*2} + \beta^{*2} + \gamma^{*2})]\right), \end{aligned} \tag{21}$$

where

$$C_1 = -2(s^8 - s^4 - 4), \qquad C_2 = 4s^2(s^4 - 2), \qquad C_3 = -4s^4, \qquad C_4 = 4s^2.$$

The most important point to note here is the emergence of mixed conjugate/non-conjugate pairs, which do not appear in the Wigner function for the second-quantized NOPA-like optical analogue (see appendix A.1). To make the behaviour of the Wigner function in the asymptotic region clearer, the parameters α, β and γ are written in polar form, $\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}\phi_\alpha}$ etc. The Wigner function thus becomes

$$W(\alpha,\beta,\gamma) = \frac{8}{\pi^3}\exp\left(\frac{1}{(s^4-4)(s^4-1)}[C_1(|\alpha|^2+|\beta|^2+|\gamma|^2) \right.$$
$$+2C_2(|\alpha||\beta|\cos(\phi_\alpha+\phi_\beta)+|\beta||\gamma|\cos(\phi_\beta+\phi_\gamma)+|\gamma||\alpha|\cos(\phi_\gamma+\phi_\alpha))$$
$$+2C_3(|\alpha||\beta|\cos(\phi_\beta-\phi_\alpha)+|\beta||\gamma|\cos(\phi_\gamma-\phi_\beta)+|\gamma||\alpha|\cos(\phi_\gamma-\phi_\alpha))$$
$$\left. +2C_4(|\alpha|^2\cos(2\phi_\alpha)+|\beta|^2\cos(2\phi_\beta)+|\gamma|^2\cos(2\phi_\gamma))]\right). \qquad (22)$$

4. CHSH inequalities and violations

The CHSH inequalities [6] are CV generalizations of the original Bell inequalities which were set up to test the scheme proposed by Einstein, Podolsky and Rosen (EPR) in 1935 [1]. Considering the measurement of an entangled pair of particles performed after they have been separated such that no classical communication channels are open when the wavefunction collapses, EPR posited that either quantum mechanics must be incomplete, with room for a hidden variable theory, or spatiotemporal locality is violated. Bell showed that hidden variables were not permitted if we preserve both the assumptions of accepted theory—specifically locality—and the probabilities predicted by quantum mechanics.

Generalized N-mode Bell inequalities—CHSH inequalities, in terms of the Bell operator expectation values B_N—exist (see for example [16]). In their bi- and tripartite form we can apply these to our regularized EPR-like states. Following [10, 13], the CHSH inequalities for the bi- and tripartite forms are the possible combinations:

$$\begin{aligned} &B_2 = \Pi(0,0)+\Pi(0,\beta)+\Pi(\alpha,0)-\Pi(\alpha,\beta),\\ &|B_2|\leqslant 2, \end{aligned} \qquad (23)$$

$$\begin{aligned} &B_3 = \Pi(0,0,\gamma)+\Pi(0,\beta,0)+\Pi(\alpha,0,0)-\Pi(\alpha,\beta,\gamma),\\ &|B_3|\leqslant 2. \end{aligned} \qquad (24)$$

In [13], the CHSH inequality constructed with the tripartite NOPA-like state (see equation (A.5)) is maximized by taking an all-imaginary substitution $\alpha=\beta=\gamma=\mathrm{i}\sqrt{J}$, where J is some distance measure. In the bipartite case (figure 1), if we look in the region $s\to 1^+$ for (8), an all-imaginary substitution obviously gives exactly the same maximum violation as NOPA with $r\to\infty$, both having a maximum value of $B_2^{\max}\approx 2.19$ [10, 13]. The figure shows clearly that the value of B_2 increases as $s\to 1^+$ and $J\to 0$.

In the tripartite case however, we must examine the wealth of other possible choices which extremise the inequality. From the form of the Wigner function in (22), however, there are some clear choices that will minimize the last term in B_3. Choosing all the phases $\phi_\alpha=\phi_\beta=\phi_\gamma=\frac{\pi}{2}$, and all the magnitudes $|\alpha|=|\beta|=|\gamma|=\sqrt{J}$, such that all the parameters are imaginary ($\mathrm{i}\sqrt{J}$), equation (22) becomes

$$\begin{aligned} &W(\mathrm{i}\sqrt{J},0,0)=W(0,\mathrm{i}\sqrt{J},0)=W(0,0,\mathrm{i}\sqrt{J})=\frac{8}{\pi^3}\exp\left(-\frac{J(s^4-s^2+2)}{(s^2+1)(s^2-2)}\right),\\ &W(\mathrm{i}\sqrt{J},\mathrm{i}\sqrt{J},\mathrm{i}\sqrt{J})=\frac{8}{\pi^3}\exp\left(-\frac{3J(s^2+2)}{s^2-2}\right). \end{aligned} \qquad (25)$$

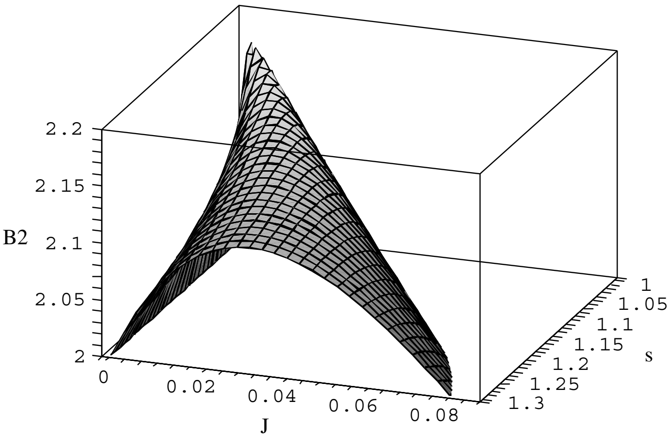

Figure 1. Plot of bipartite s-modified CHSH, with an all-imaginary choice for α and β. Reaches a maximum value of ≈ 2.19 as $s \to 1$ and $J \to 0$. Note that this is equivalent to the NOPA case.

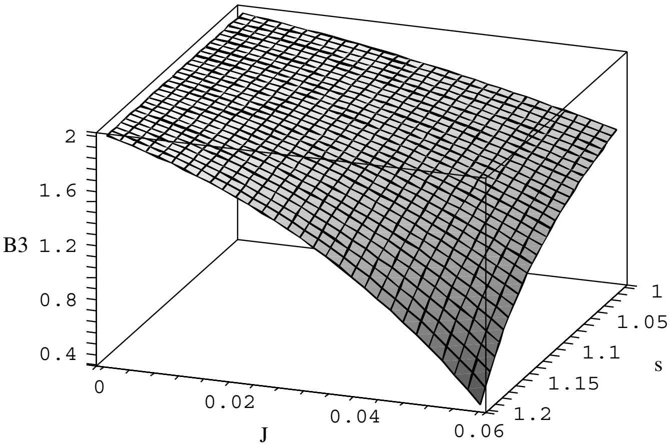

Figure 2. Tripartite s-modified CHSH. With an all-imaginary choice for α, β and γ, B_3 never reaches a value greater than 2 as $s \to 1$.

Consequently B_3 is (from (24)):

$$B_3 = 3\exp\left\{-\frac{J(s^4 - s^2 + 2)}{(s^2+1)(s^2-2)}\right\} - \exp\left\{-\frac{3J(s^2+2)}{(s^2-2)}\right\}. \tag{26}$$

In the region $s \to 1^+$, B_3 never reaches a value greater than 2 (figure 2). A violation corresponding to the EPR limit $s \to 1^+$ can be found by making the choice $\alpha = -\beta = -\sqrt{J}$; $\gamma = 0$, for which (22) gives

$$\begin{aligned} &W(-\sqrt{J}, 0, 0) = W(0, \sqrt{J}, 0) = \exp\left\{-\frac{J(s^4+s^2+2)}{(s^2-1)(s^2+2)}\right\}, \\ &W(-\sqrt{J}, \sqrt{J}, 0) = \frac{8}{\pi^3}\exp\left(-\frac{2J(s^2+1)}{(s-1)(s+1)}\right), \\ &W(0,0,0) = 1, \end{aligned} \tag{27}$$

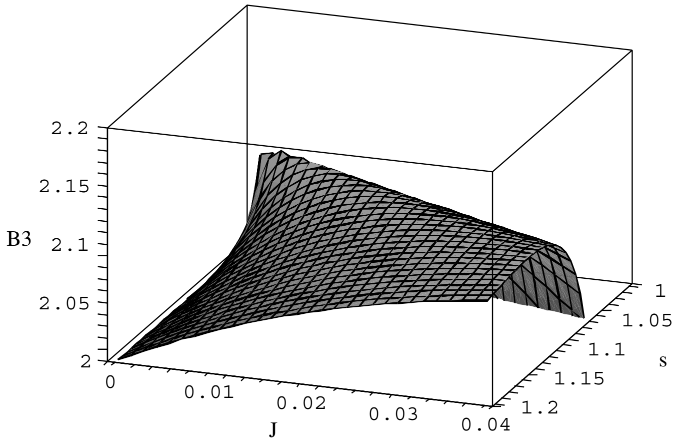

Figure 3. Tripartite s-modified CHSH. With $\alpha = -\beta = -\sqrt{J}$, $\gamma = 0$, B_3 reaches a maximum value of ≈ 2.09 as $s \to 1^+$ and $J \to 0$.

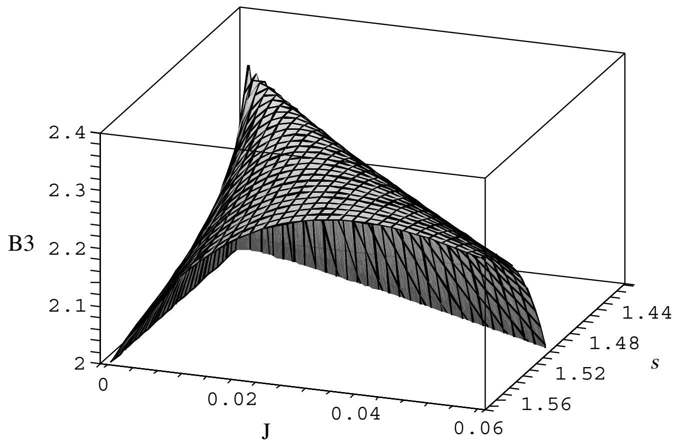

Figure 4. Tripartite s-modified CHSH. With an all imaginary choice for α, β and γ, B_3 reaches a maximum value of ≈ 2.32 as $s \to \sqrt{2^+}$ and $J \to 0$.

and B_3 becomes (figure 3)

$$B_3 = 1 + 2\exp\left\{-\frac{J(s^4+s^2+2)}{(s^2-1)(s^2+2)}\right\} - \exp\left\{-\frac{2J(s^2+1)}{(s^2-1)}\right\}. \tag{28}$$

As $s \to 1^+, J \to 0$, the maximum value is $B_3^{\max} \approx 2.09$, which can be checked both analytically and numerically.

However, what is more interesting still is exploring an auxiliary regime of the regulator, $s \to \sqrt{2^+}$, in equation (26). This is shown in figure 4. Analytically, we can approximate the maximum to the lowest order in $s - \sqrt{2} = \epsilon$ by writing $B_3 = 3x - x^\lambda$, where $x = \exp(-4J/3\epsilon^2)$, and $\lambda = 9$, with maximum

$$B_3^{\max} \cong (\lambda - 1)\left(\frac{3}{\lambda}\right)^{\frac{\lambda}{\lambda-1}} \cong 2.32 \tag{29}$$

at $x = \left(\frac{3}{\lambda}\right)^{\frac{1}{\lambda-1}}$. This can be confirmed numerically for $s \to \sqrt{2^+}$, $J \to 0$. The values of $B_3^{\max}$ correspond exactly to those calculated for the experimentally verified NOPA-like states, whose maximization as $r \to \infty$ is also governed by (29).

5. Discussion

In this paper we have analysed tripartite CV entangled states which are natural generalizations of the classic bipartite EPR-type states (for two systems with canonical variables X_1, P_1, X_2, P_2). Given the necessity of working with normalizable states which still approximate the ideal EPR-type limit for practical implementation of CHSH inequalities, we examined a family of such regulated states parameterized by a regulating parameter s. This family of states was compared with those relating to multipartite NOPA-like states. The NOPA states have been shown to manifest CHSH violations, and have the advantage of being directly accessible by experiment via standard quantum optics protocols such as multiparametric heterodyne detection techniques and beam splitter operations. However, as an extension of a direct transcription of the EPR paradox, this new family of regularized states provides an alternative, systematic description of the approach to the ideal EPR states for relative variables.

By finding expressions for the eigenstates of the regularized tripartite CV EPR-like states it became apparent that there are two regimes of the regularization parameter in which these states become singular: in one case ($s \to 1$) we have a singular eigenstate of the relative coordinates while remaining squeezed in the total momentum; in the other, $s \to \sqrt{2}$ limit we have a singular eigenstate of the total momentum, but squeezed in the relative coordinates. In these two regimes we have explored CHSH inequalities via Wigner functions regarded as expectation values of displaced parity operators. Violations of the tripartite CHSH bound ($B_3 \leqslant 2$) are established analytically and numerically, with $B_3 \cong 2.09$ in the canonical regime ($s \to 1^+$), as well as $B_3 \cong 2.32$ in the auxiliary regime ($s \to \sqrt{2^+}$).

Related tripartite entangled states have recently been constructed by Fan [26]. Although these states are also accessible by standard quantum optics techniques, they are not true generalizations of 'EPR' states. In this case, while they diagonalize one centre-of-mass variable (for example, $X_1 + X_2 + X_3$), they are *coherent* states [27] of the remaining relative Jacobi observables (that is, they diagonalize their annihilation mode operators $\mathfrak{a}$, $\mathfrak{b}$ in contrast to the $s \to \sqrt{2}$ limit of our EPR-type tripartite states, which as stated above turn out to be *squeezed* states of these relative degrees of freedom (eigenstates of a linear combination $\frac{1}{\sqrt{3}}(2\mathfrak{a}+\mathfrak{a}^\dagger)$, $\frac{1}{\sqrt{3}}(2\mathfrak{b}+\mathfrak{b}^\dagger)$ in the relative mode operators, with the value $\frac{1}{2}\ln 3$ for the squeezing parameter). In the case of the tripartite entangled states of [26], no regularization has been given. A construction of true multipartite ideal EPR states has, however, been provided in [18, 19], with a second-quantized form for the tripartite state (A.19), which may be compared with the form for the NOPA-like state (A.4). Although the Wigner functions for the tripartite NOPA-like states show peaks at zeroes of $X_i - X_j$ and $P_i + P_j$ for all distinct pairs i, j [11], which does not appear to be consistent with simultaneous diagonalization of commuting observables, it can be inferred from the agreement of (A.19) and (A.4) that indeed in the infinite squeezing limit, $\tanh(r) = 1$, and with relative parameters equal to zero, the NOPA-like state does again tend to the ideal EPR state.

Since the NOPA-like states are constructed with a view to experimental realizability, and, in the bipartite case, to manufacture the specific properties of the Wigner function, this new suggestion for a regularization stemming from a direct transcription of the EPR paradox in terms of the simultaneous diagonalization of commuting observables could be seen as a more general or fundamental description. It also considers in more detail the specific instance of tripartite EPR-type states, compared to the comprehensive [19] which finds n-partite representations of entangled states through their Gaussian-form completeness relation without exploring regularizations and Wigner function properties. As the proposed EPR-type regularized state produces a different Wigner function from the NOPA-type, with two singular limits, this paper's proposed regularization may potentially suggest that alternative experimental ways to achieve the violations of the CHSH inequalities are possible. For a review of Gaussian states, and discussions of the realizability of entangled states, we refer to [21, 28, 29], and references therein. It will be worth investigating the full extent of the constraints placed on the choices of displacement parameters entailed by the shift in η (see appendix A.2). The current discussion might also easily be extended to include a presentation of the alternative bipartite starting point of conjugate variable choice X_1+X_2 and P_1-P_2. [17] discusses the canonical combinations for any number of modes, but in our case it is reasonable to assume that an N-partite generalization would be of the form $\left[\exp\left(\frac{1}{s^2}\left(\sum_{i<j} a_i^\dagger a_j^\dagger\right)\right)\right]$. We would also expect that these states would admit standard completeness relations in the singular cases.

In conclusion, we have presented a rigorous extension of Fan and Klauder's general EPR-like states to the regularized tripartite CV case for relative variables, and highlighted the connection to current quantum optics implementations. The CHSH inequalities constructed with component Wigner functions for this case show significant violation of the classical bound, and the different choices of regularization parameter making the state singular illustrate an interesting new feature of the structure of generalized CV EPR-like states.

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Appendix A. Deriving the tripartite NOPA and EPR-like Wigner functions

In this appendix the tripartite NOPA and $|\eta, \eta', \eta''\rangle_s$ Wigner functions are derived for comparison.

A.1. Second-quantized form and Wigner function of tripartite NOPA-like state

Applying two phase-free beamsplitters at specified angles acting on one momentum squeezed state and two position squeezed states of mode 1, 2 and 3 respectively, [30] states that the tripartite NOPA-like states can be derived from the following expression:

$$|\mathrm{NOPA}^{(3)}\rangle = B_{23}\left(\frac{\pi}{4}\right) B_{12}\left(\arccos\frac{1}{\sqrt{3}}\right) \times \exp\left(\frac{r}{2}(a^2 - a^{\dagger 2})\right) \exp\left(\frac{-r}{2}(b^2 - b^{\dagger 2})\right) \exp\left(\frac{-r}{2}(c^2 - c^{\dagger 2})\right) |000\rangle. \tag{A.1}$$

We can therefore use the following formula quoted in [31][4] for the squeezing operator $S(z)$ (where $z = \mathrm{e}^{\mathrm{i}\theta}$) with the Baker–Campbell–Hausdorff (BCH) relation:

$$\begin{aligned} S(z) &= \exp\left[\tfrac{1}{2}(za^{\dagger 2} - \bar{z}a^2)\right] \\ &= \exp\left[\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta}\tanh r)a^{\dagger 2}\right]\exp\left[-2(\ln\cosh r)\left(\tfrac{1}{2}a^\dagger a + \tfrac{1}{4}\right)\right]\exp\left[-\tfrac{1}{2}(\mathrm{e}^{-\mathrm{i}\theta}\tanh r)a^2\right]. \end{aligned} \tag{A.2}$$

The following beamsplitter operation[5] can then be applied, where θ here refers to the angles $\pi/4$ and $\arccos(1/\sqrt{3})$ for the B_{23} and B_{12} splitters respectively:

$$B_{ab}(\theta) : \begin{cases} a \to a\cos\theta + b\sin\theta \\ b \to -a\sin\theta + b\cos\theta \end{cases} \tag{A.3}$$

and normalizing, the tripartite NOPA is expressible in second-quantized form as

$$\begin{aligned} |\mathrm{NOPA}^{(3)}\rangle &= (1 - \tanh^2(r))^{3/4} \\ &\quad\times \exp\left(-\tfrac{1}{6}\tanh r(a^{\dagger 2} + b^{\dagger 2} + c^{\dagger 2}) + \tfrac{2}{3}\tanh r(b^\dagger c^\dagger + a^\dagger b^\dagger + a^\dagger c^\dagger)\right)|000\rangle. \end{aligned} \tag{A.4}$$

Using this state to derive the Wigner function of the form (17), the complex exponential that is produced may be rearranged using formula (A.14). The tripartite NOPA Wigner function then becomes

$$\begin{aligned} W_{\mathrm{NOPA}} &= \left(\frac{2}{\pi}\right)^3 \exp\left\{\left(2 - \frac{4}{1 - \tanh^2 r}\right)(|\alpha|^2 + |\beta|^2 + |\gamma|^2)\right. \\ &\quad - \frac{2\tanh r}{3(1 - \tanh^2 r)}(\alpha^{*2} + \beta^{*2} + \gamma^{*2} + \alpha^2 + \beta^2 + \gamma^2) \\ &\quad \left. + \frac{8\tanh r}{3(1 - \tanh^2 r)}(\alpha\beta + \beta\gamma + \gamma\alpha + \alpha^*\beta^* + \beta^*\gamma^* + \gamma^*\alpha^*)\right\} \\ &= \frac{8}{\pi^3}\exp\left\{(-2\cosh(2r))(|\alpha|^2 + |\beta|^2 + |\gamma|^2)\right. \\ &\quad - \frac{1}{3}\sinh(2r)(\alpha^{*2} + \beta^{*2} + \gamma^{*2} + \alpha^2 + \beta^2 + \gamma^2) \\ &\quad \left. + \frac{4}{3}\sinh(2r)(\alpha\beta + \beta\gamma + \gamma\alpha + \alpha^*\beta^* + \beta^*\gamma^* + \gamma^*\alpha^*)\right\}. \end{aligned} \tag{A.5}$$

This is the result quoted in [13], and further explication can be found in that paper. This function should be compared with the Wigner function of our regularized tripartite EPR-like state, $|\eta, \eta', \eta''\rangle_s$. Further details of that derivation are given in appendix A.3.

A.2. Derivation of conditions for $\eta = 0$

In the interest of brevity, the conditions for shifting the η parameters are shown below for the bipartite case. However, the analysis extends in an obvious way to the tripartite case. After BCH and anti-normal ordering, the bipartite Wigner function (tripartite given in equation (20))

[4] Note the misprint in the sign of the last exponential in [31]; see [32].

[5] An additional overall relative sign (180° phase shift) between the two modes has been omitted; see for example [33].

becomes

$$W(\alpha,\beta) = {}_s\langle\eta,\eta'|\underbrace{\mathrm{e}^{2|\alpha|^2}\,\mathrm{e}^{2|\beta|^2}\,\mathrm{e}^{-2\alpha^* a}\,\mathrm{e}^{-2\beta^* b}\,\mathrm{e}^{2\alpha a^\dagger}\,\mathrm{e}^{2\beta b^\dagger}}_{F(\alpha,\beta)}(-1)^{n_a+n_b}|\eta,\eta'\rangle_s$$
$$= \langle 00|\exp\left(-\frac{1}{4s^2}|\eta|^2 - \frac{1}{4s^2}|\eta'|^2 + \frac{1}{s}\eta^* a + \frac{1}{s}\eta'^* b + \frac{1}{s^2}ab\right)F(\alpha,\beta)$$
$$\times\exp\left(-\frac{1}{4s^2}|\eta|^2 - \frac{1}{4s^2}|\eta'|^2 - \frac{1}{s}\eta a^\dagger - \frac{1}{s}\eta' b^\dagger + \frac{1}{s^2}a^\dagger b^\dagger\right)|00\rangle. \quad \text{(A.6)}$$

We then make the generic substitutions

$$\alpha = \alpha' + A(\eta,s), \qquad \beta = \beta' + B(\eta',s), \quad \text{(A.7)}$$

into $F(\alpha,\beta)$. To find the expressions for $A(\eta,s)$ and $B(\eta',s)$ that will allow us to set $\eta = \eta'(=\eta'') = 0$, we solve the following:

$$\begin{aligned} 2\alpha' A^* + 2\alpha'^* A + 2|A|^2 - 2A^* a + 2A a^\dagger &= \frac{1}{2s^2}|\eta|^2 - \frac{1}{s}\eta^* a + \frac{1}{s}\eta a^\dagger, \\ 2\beta' B^* + 2\beta'^* B + 2|B|^2 - 2B^* b + 2B b^\dagger &= \frac{1}{2s^2}|\eta'|^2 - \frac{1}{s}\eta'^* b + \frac{1}{s}\eta' b^\dagger. \end{aligned} \quad \text{(A.8)}$$

Thus we can see that, if we allow the constraints

$$\frac{\alpha'\eta^*}{s} + \frac{\alpha'^*\eta}{s} = 0, \qquad \frac{\beta'\eta'^*}{s} + \frac{\beta'^*\eta'}{s} = 0, \quad \text{(A.9)}$$

(i.e. α' real and η imaginary or vice versa), then the expressions for $A(\eta,s)$ and $B(\eta',s)$ become

$$\begin{aligned} A(\eta,s) &= \frac{\eta}{2s}, \qquad A^*(\eta,s) = \frac{\eta^*}{2s}, \\ B(\eta',s) &= \frac{\eta'}{2s}, \qquad B^*(\eta',s) = \frac{\eta'^*}{2s}. \end{aligned} \quad \text{(A.10)}$$

Therefore, up to a factor, taking $\eta = \eta'(=\eta'') = 0$ corresponds to a shift in the parameters of the displacement operators $\alpha' = \alpha - \frac{\eta}{2s}$, $\beta' = \beta - \frac{\eta'}{2s}$ (and $\gamma' = \gamma - \frac{\eta''}{2s}$ in the tripartite case). For the tripartite Wigner function as used in section (3), this can be expressed as

$$W_{\eta,\eta',\eta''}(\alpha',\beta',\gamma') = E(\alpha',\beta',\gamma',\eta,\eta',\eta'')W_{0,0,0}(\alpha,\beta,\gamma) \quad \text{(A.11)}$$

$$E(\alpha',\beta',\gamma',\eta,\eta',\eta'') = \exp\left(\frac{1}{s}(\alpha'\eta^* + \alpha'^*\eta + \beta'\eta'^* + \beta'^*\eta' + \gamma'\eta''^* + \gamma'^*\eta'')\right) \quad \text{(A.12)}$$

A.3. Details of derivation of tripartite $|\eta,\eta',\eta''\rangle_s$ Wigner function

The second-quantized EPR-like state is expressed as (9). This is used to find the Wigner function in the form of (20). By commuting mode operators with the parity operator and rearranging using BCH identities, the expression becomes, in anti-normal ordered form:

$$W = \left(\frac{2}{\pi}\right)^3 N_3^2\,\mathrm{e}^{-\frac{1}{2s^2}|\eta|^2 - \frac{1}{2s^2}|\eta'|^2 - \frac{1}{2s^2}|\eta''|^2}$$
$$\times\langle 000|\exp\left(\frac{1}{s}(\eta^* a + \eta'^* b + \eta''^* c) + \frac{1}{s^2}(ab + ac + bc)\right)$$
$$\times\,\mathrm{e}^{2|\alpha|^2}\,\mathrm{e}^{2|\beta|^2}\,\mathrm{e}^{2|\gamma|^2}\,\mathrm{e}^{-2\alpha^* a}\,\mathrm{e}^{-2\beta^* b}\,\mathrm{e}^{-2\gamma^* c}\,\mathrm{e}^{2\alpha a^\dagger}\,\mathrm{e}^{2\beta b^\dagger}\,\mathrm{e}^{2\gamma c^\dagger}$$
$$\times\exp\left(\frac{1}{s}(\eta a^\dagger + \eta' b^\dagger + \eta'' c^\dagger) + \frac{1}{s^2}(a^\dagger b^\dagger + a^\dagger c^\dagger + b^\dagger c^\dagger)\right)|000\rangle. \quad \text{(A.13)}$$

In anti-normal ordered form we may insert a complete set of coherent states $\int |u, v, w\rangle\langle u, v, w| \frac{d^2u\, d^2v\, d^2w}{\pi}$ such that we may rearrange the exponential according to the formula [34, 35]:

$$\int \prod_i^n \left[\frac{d^2 z_i}{\pi}\right] \exp\left(-\frac{1}{2}(z, z^*)\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} z \\ z^*\end{pmatrix} + (\mu, \nu^*)\begin{pmatrix} z \\ z^*\end{pmatrix}\right)$$
$$= \left[\det\begin{pmatrix} C & D \\ A & B\end{pmatrix}\right]^{-\frac{1}{2}} \exp\left[\frac{1}{2}(\mu, \nu^*)\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}\begin{pmatrix} \mu \\ \nu^*\end{pmatrix}\right]$$
$$= \left[\det\begin{pmatrix} C & D \\ A & B\end{pmatrix}\right]^{-\frac{1}{2}} \exp\left[\frac{1}{2}(\mu, \nu^*)\begin{pmatrix} C & D \\ A & B\end{pmatrix}^{-1}\begin{pmatrix} \nu^* \\ \mu\end{pmatrix}\right], \tag{A.14}$$

where matrices A and D must be symmetrical, and $C = B^T$. In this instance

$$(z, z^*) = (u, v, w, u^*, v^*, w^*),$$
$$(\mu, \nu^*) = \left(\frac{1}{s}\eta^* - 2\alpha^*, \frac{1}{s}\eta'^* - 2\beta^*, \frac{1}{s}\eta''^* - 2\gamma^*, -\frac{1}{s}\eta + 2\alpha, -\frac{1}{s}\eta' + 2\beta, -\frac{1}{s}\eta'' + 2\gamma\right), \tag{A.15}$$

and we have

$$\begin{pmatrix} C & D \\ A & B\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{s^2} & -\frac{1}{s^2} \\ 0 & 1 & 0 & -\frac{1}{s^2} & 0 & -\frac{1}{s^2} \\ 0 & 0 & 1 & -\frac{1}{s^2} & -\frac{1}{s^2} & 0 \\ 0 & -\frac{1}{s^2} & -\frac{1}{s^2} & 1 & 0 & 0 \\ -\frac{1}{s^2} & 0 & -\frac{1}{s^2} & 0 & 1 & 0 \\ -\frac{1}{s^2} & -\frac{1}{s^2} & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.16}$$

with inverse

$$\begin{pmatrix} C & D \\ A & B\end{pmatrix}^{-1} = \frac{s^4}{(s^4-4)(s^4-1)} \begin{pmatrix} s^4-3 & 1 & 1 & 2s^{-2} & \frac{s^4-2}{s^2} & \frac{s^4-2}{s^2} \\ 1 & s^4-3 & 1 & \frac{s^4-2}{s^2} & 2s^{-2} & \frac{s^4-2}{s^2} \\ 1 & 1 & s^4-3 & \frac{s^4-2}{s^2} & \frac{s^4-2}{s^2} & 2s^{-2} \\ 2s^{-2} & \frac{s^4-2}{s^2} & \frac{s^4-2}{s^2} & s^4-3 & 1 & 1 \\ \frac{s^4-2}{s^2} & 2s^{-2} & \frac{s^4-2}{s^2} & 1 & s^4-3 & 1 \\ \frac{s^4-2}{s^2} & \frac{s^4-2}{s^2} & 2s^{-2} & 1 & 1 & s^4-3 \end{pmatrix}. \tag{A.17}$$

Note also that

$$\left[\det\begin{pmatrix} C & D \\ A & B\end{pmatrix}\right]^{-\frac{1}{2}} = [(s^{12} - 6s^7 + 9s^4 - 4)/s^{12}]^{-\frac{1}{2}} = \frac{1}{N_3^2}, \tag{A.18}$$

such that the N_3^2 cancel in the Wigner function.

From the argument in appendix A.2, we assume that $\eta = \eta' = \eta'' = 0$ unless otherwise specified, and continue to use $W(\alpha, \beta, \gamma)$. This gives equation (21), which may now easily be compared with the Wigner function derived for the NOPA-like case (equation (A.5)). Further discussion of similar manipulations of the Wigner function can be found in [36].

A.4. Tripartite entangled state from [19]

Equation (27) in [19] provides the ideal EPR state for the tripartite entangled state:

$$
\begin{aligned}
|p, \xi_2, \xi_3\rangle &= \frac{1}{\sqrt{3}\pi^{\frac{3}{4}}} \exp\left[A + \frac{\mathrm{i}\sqrt{2}p}{3}\sum_{i=1}^{3} a_i^\dagger + \frac{\sqrt{2}\xi_2}{3}\left(a_1^\dagger - 2a_2^\dagger + a_3^\dagger\right)\right.\\
&\qquad \left. + \frac{\sqrt{2}\xi_3}{3}\left(a_1^\dagger + a_2^\dagger - 2a_3^\dagger\right) + S^\dagger\right]|000\rangle,\\
A &\equiv -\frac{p^2}{6} - \frac{1}{3}\left(\xi_2^2 + \xi_3^2 - \xi_2\xi_3\right),\\
S &\equiv \frac{2}{3}\sum_{i<j=1}^{3} a_i a_j - \frac{1}{6}\sum_{i=1}^{3} a_i^2.
\end{aligned}
\tag{A.19}
$$

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